

Rigorous Theory of the Quantum Electromagnetic Field in the Presence of Quantum Sources

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We give a rigorous description of a model of the quantized electromagnetic field interacting with quantized current fields. In the special case of classical currents our results agree with common knowledge about the problem. A toy model of a quantum current is studied as well.

KEY WORDS: quantized electromagnetic field; classical current; quantum current; covariance systems.

1. INTRODUCTION

The traditional approach to quantum electrodynamics starts with the theory of free photons, respectively free electrons. Next, interactions are introduced. They are treated using scattering theory, making the assumption that particles are free in the large time limit. It is well known that the resulting theory suffers from intrinsic difficulties, such as infrared and ultraviolet divergences. Here we show how to describe in a rigorous manner an electromagnetic (e.m.) field interacting with quantized current fields. In principle, these fields might be generated by Dirac electron fields. However, we prefer to avoid the difficulties of fully quantized electrodynamics by delaying the rigorous formulation of electron fields to a forthcoming paper. The integration of both models into a mathematically acceptable theory of QED remains an open problem.

Important for the rigorous formulation of the problem at hand is the choice of method for quantizing the e.m. field. As shown quite some time ago (Carey, Gaffney, and Hurst, 1977), the method of Fermi can be made rigorous. The essence of Carey, Gaffney, and Hurst (1977) has been picked up in Kuna and Naudts (2002) and was used to give an elegant formulation of quantized free e.m. fields using the formalism of covariance systems (Naudts and Kuna, 2001). In particular, free-field operators are smeared out using test functions satisfying the continuity equation.

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By doing so they become gauge independent (Kibble, 1968) and the difficult problem of gauge covariance of the quantized theory is avoided.

Next a model is needed for the current fields. Given any such fields one can construct vector potential fields by integrating the currents with the Green's function of the d'Alembert equation. These vector potential fields can be described in a similar way as the free-field potentials. This is the basis for the present paper. The quantized e.m. fields and the currents are described simultaneously by a single covariance system. An essential property of such a covariance system is that only minimal information is needed about the properties of the two subsystems and of their interaction, because such information is encoded in the states of the system, which are determined by their correlation function. To illustrate this point we discuss states describing quantized e.m. fields interacting with classical currents and show that in this case we recover known results. But we discuss also states describing genuine interaction with arbitrary current fields.

The structure of the paper is as follows. Sections 2, 3, and 4 serve as a technical introduction. Our basic ansatz is made in Section 5. Section 6 contains two propositions exploring the consequences of this ansatz. Sections 7 and 8 show that our results are in agreement with results of the standard approach. In Section 9 we introduce a simplified model of quantum currents. The physical interpretation of radiation fields of this model follows in Section 10. Finally, Section 11 draws some conclusions. Two Appendices contain some further technical matters.

2. TEST FUNCTIONS

In photon theory a number of pitfalls have to be avoided and some choices have to be made. To begin with, it is well known that the field operators $\hat{A}_\mu(q)$ do only exist in a distributional sense. Therefore, smeared-out operators are defined by

$$\hat{A}(f) = \int_{\mathbb{R}^4} dq f^\mu(q) \hat{A}_\mu(q) \quad (1)$$

The $f_\mu(q)$ are test functions reflecting the experimental inaccuracy to select a single point of Minkowski space. The Fourier-transformed function $\tilde{f}_\mu(k)$ satisfies

$$f_\mu(q) = (2\pi)^{-2} \int_{\mathbb{R}^4} dk \tilde{f}_\mu(k) e^{-iq^\nu k_\nu} \quad (2)$$

From the reality of $f_\mu(q)$ follows that $\overline{\tilde{f}_\mu(k)} = \tilde{f}_\mu(-k)$.

Following Kibble (1968) and Kuna and Naudts (2002), we assume that the test functions satisfy the Fourier-transformed continuity equation

$$k^\mu \tilde{f}_\mu(k) = 0 \quad (3)$$

A justification is given in Appendix A. This assumption is essential in the present formalism to control problems with gauge invariance of the theory.

In what follows the space of test functions f_μ is denoted G . It consists of real functions $f_\mu(q)$ whose Fourier transform satisfies (3). For technical reasons, we assume that the Fourier-transformed functions are continuous and vanish outside a bounded region.

3. CLASSICAL WAVE FUNCTIONS

It is tradition to introduce a complex Hilbert space of so-called classical wave functions of the photon. The space G is only a real space, because of the condition that $f_\mu(q)$ is real, while the classical wave functions form a complex pre-Hilbert space H . These are square integrable complex functions $\phi_\mu(\mathbf{k})$, defined for \mathbf{k} in \mathbb{R}^3 , satisfying

$$|\mathbf{k}| \phi_0(\mathbf{k}) = \sum_{\alpha=1}^3 k_\alpha \phi_\alpha(\mathbf{k}) \quad (4)$$

With each test function $f_\mu(q)$ corresponds a classical wave function $\phi_\mu(\mathbf{k})$ by the relation

$$\phi_\mu(\mathbf{k}) = \sqrt{2\pi} \tilde{f}_\mu(|\mathbf{k}|, \mathbf{k}), \quad \mathbf{k} \in \mathbb{R}^3 \quad (5)$$

The (degenerate) scalar product for classical wave functions is given by

$$\langle \phi | \psi \rangle = - \int_{\mathbb{R}^3} d\mathbf{k} \frac{1}{2|\mathbf{k}|} \overline{\phi^\mu(\mathbf{k})} \psi_\mu(\mathbf{k}) \quad (6)$$

Positivity of this scalar product follows because the classical wave functions satisfy (4). Indeed, one has

$$\begin{aligned} \langle \phi | \phi \rangle &= - \int_{\mathbb{R}^3} d\mathbf{k} \frac{1}{2|\mathbf{k}|^3} \left(|\mathbf{k}|^2 |\phi_0(\mathbf{k})|^2 - |\mathbf{k}|^2 \sum_{\alpha=1}^3 |\phi_\alpha(\mathbf{k})|^2 \right) \\ &= \int_{\mathbb{R}^3} d\mathbf{k} \frac{1}{2|\mathbf{k}|^3} \sum_{\alpha,\beta=1}^3 [\delta_{\alpha\beta} |\mathbf{k}|^2 - k_\alpha k_\beta] \overline{\phi_\alpha(k)} \phi_\beta(k) \\ &\geq 0 \end{aligned} \quad (7)$$

The latter holds because the matrix with elements $\delta_{\alpha\beta} |\mathbf{k}|^2 - k_\alpha k_\beta$ is positive definite for all values of \mathbf{k} . We conclude that the classical wave functions form a pre-Hilbert space H .

4. CORRELATION FUNCTION DESCRIPTION OF E.M. FIELDS

The smeared-out field operators, which will be constructed later on, satisfy the canonical commutation relations

$$[\hat{A}(f), \hat{A}(g)]_- = -2i \operatorname{Im}(\phi | \psi) \quad (8)$$

where ϕ and ψ are the classical wave functions determined by f and g , respectively. The displacement operators are defined by

$$\hat{W}(f) = \exp(i \hat{A}(f)) \quad (9)$$

They satisfy the Weyl form of commutation relations

$$\hat{W}(f)\hat{W}(g) = e^{i \operatorname{Im}(\phi|\psi)} \hat{W}(f+g) \quad (10)$$

and generate an algebra which is not the algebra of canonical commutation relations (Petz, 1990) because the symplectic form

$$f, g \rightarrow \operatorname{Im} \langle \phi | \psi \rangle \quad (11)$$

is clearly degenerate.

We need several distinct representations of this algebra. Free photon fields are described by field operators in Fock space. On the other hand, if the presence of an external current produces an infrared divergency, then a representation is needed which differs from the Fock representation. The obvious way to handle such a situation is by means of correlation functions. They determine the Hilbert space representation uniquely by means of the GNS representation theorem.

Instead of working with mathematical states of the C^* -algebra of canonical commutation relations (Petz, 1990) we work with the correlation function formalism of Naudts and Kuna (2001). The basic quantity is the two-point correlation function $\mathcal{F}(f, g)$, defined for any pair of test functions f and g in G . In a Hilbert space representation with state vector Ω , it has the following meaning

$$\mathcal{F}(f, g) = \langle \hat{W}(g)^* \Omega | \hat{W}(f)^* \Omega \rangle \quad (12)$$

The scalar product between two vectors Φ and Ψ of the Hilbert space is denoted $\langle \Psi | \Phi \rangle$ and is complex linear in Φ , antilinear in Ψ .

From (12) it is obvious that the correlation function $\mathcal{F}(f, g)$ is nothing but the inner product between two coherent states, one with state vector $\hat{W}(f)^* \Omega$, the other with state vector $\hat{W}(g)^* \Omega$. The characterizing properties of correlation functions are in the present context

- (normalization) $\mathcal{F}(0, 0) = 1$;
- (positivity) $\sum_{j,k} \lambda_j \lambda_k^* \mathcal{F}(f_j, f_k) \geq 0$ for all finite sequences of complex numbers $\lambda_1, \dots, \lambda_n$ and of elements f_1, \dots, f_n of G .

- (covariance) There exists a symplectic form σ over G such that

$$\mathcal{F}(f + h, g + h) = e^{i\sigma(f-g, h)} \mathcal{F}(f, g) \tag{13}$$

holds for all f, g, h in G .

The general definition of states of a covariance system contains also a requirement of continuity. However, the additive group of test functions $G, +$ is equipped with the discrete topology. Hence, continuity of the map $f, g \rightarrow \mathcal{F}(f, g)$ is always satisfied.

In particular, the correlation function for the vacuum state of the free photon field is given by Kuna and Naudts (2002)

$$\begin{aligned} \mathcal{F}(f, g) &= e^{(\phi|\psi)} e^{-(1/2)(\phi|\psi)} e^{-(1/2)(\psi|\psi)} \\ &= e^{i\sigma(f, g)} e^{-(1/2)s(f-g, f-g)} \end{aligned} \tag{14}$$

with

$$\begin{aligned} \sigma(f, g) &= \text{Im}\langle \phi | \psi \rangle \\ s(f, g) &= \text{Re}\langle \phi | \psi \rangle \end{aligned} \tag{15}$$

Here, ϕ and ψ are the classical wave functions determined by f and g , respectively. Expression (14) satisfies all requirements for being a correlation function—see Appendix B. By the generalized GNS theorem (Naudts and Kuna, 2001) there exists a projective representation $\hat{W}(f)$ of the group G in a Hilbert space \mathcal{H} , and a vector Ω in \mathcal{H} , such that (12) holds.

5. DESCRIBING QUANTIZED CURRENTS

The currents $\hat{j}_\mu(q)$, given as operator-valued distributions in a Hilbert space H , can be used to construct vector potentials $\hat{A}_\mu^j(q)$ by

$$\hat{A}_\mu^j(q) = - \int dq' \Delta_G(q' - q) \hat{j}_\mu(q) \tag{16}$$

where $\Delta_G(q)$ is a Green’s function of the d’Alembert equation, i.e., is a solution of

$$\square_q \Delta_G(q) = -\delta^4(q) \tag{17}$$

One can take $\Delta_G(q)$ equal to Feynman’s propagator for massless bosons (47). The formal equation

$$\square_q \hat{A}_\mu^j(q) = \hat{j}_\mu(q) \tag{18}$$

is satisfied by construction. After smearing out with test functions one obtains

$$\hat{A}^j(f) = - \int dq \int dq' f^\mu(q) \Delta_G(q' - q) \hat{j}_\mu(q') \tag{19}$$

Given these operators one can reconstruct the current fields by applying the d'Alembert operator. Indeed, introduce the notation

$$\tau_a f_\mu(q) = f_\mu(q - a) \tag{20}$$

Then, one has formally

$$\begin{aligned} \square_a \hat{A}^j(\tau_a f) &= \int dq f^\mu(q - a) \hat{j}_\mu(q) \\ &\equiv \hat{j}(\tau_a f) \end{aligned} \tag{21}$$

Note that the free-field operators $\hat{A}(f)$ satisfy the homogeneous equation

$$\square_a \hat{A}(\tau_a f) = 0 \tag{22}$$

As a consequence, two free-field operators $\hat{A}(f)$ and $\hat{A}(g)$ are equal if they determine the same classical wave function (modulo the null space of H). This property is in general *not* true for the operators $\hat{A}^j(f)$. Nevertheless one can use the formalism of covariance systems to describe these $\hat{A}^j(f)$ in a similar way as for describing free-field operators. Introduce the notation

$$\hat{W}^j(f) = \exp(i \hat{A}^j(f)) \tag{23}$$

and assume that a correlation function $\mathcal{F}^j(f, g)$ is given such that in the GNS representation one has

$$\mathcal{F}^j(f, g) = \langle \hat{W}^j(g)^* \Omega \mid \hat{W}^j(f)^* \Omega \rangle \tag{24}$$

The current operators $\hat{j}(f)$ are fully specified by this correlation function. Examples of such functions are given below.

6. INTERACTING FIELDS

Let us now construct an interacting field operator $\hat{A}_\mu^1(q)$ which is the sum of the free-field operator $\hat{A}_\mu(q)$ and of the field $\hat{A}_\mu^j(q)$ produced by the current. The latter two are described by the correlation functions $\mathcal{F}(f, g)$, respectively $\mathcal{F}^j(f, g)$. These have to be combined into a single correlation function $\mathcal{F}^1(f, g)$, the GNS representation of which contains Weyl operators satisfying

$$\hat{W}^1(f) = \exp(i \hat{A}^1(f)) = \exp(i(\hat{A}(f) + \hat{A}^j(f))) \tag{25}$$

An easy way to produce correlation functions with the desired properties starts from correlation functions $\mathcal{F}^\times(f, f'; g, g')$ of the covariance system with group $G \times G$. By taking the diagonal of such a function, one obtains a correlation

function of the covariance system with group G

$$\mathcal{F}^1(f, g) = \mathcal{F}^\times(f, f; g, g) \quad (26)$$

The GNS representation induced by $\mathcal{F}^1(f, g)$ can be obtained from that induced by $\mathcal{F}^\times(f, f; g, g)$.

The simplest class of correlation functions of the product system consists of functions of the form

$$\begin{aligned} \mathcal{F}^\times(f, f'; g, g') &= \exp(i\sigma^\times(f, f'; g, g')) \\ &\times \exp\left(-\frac{1}{2}s^\times(f - g, f' - g'; f - g, f' - g')\right) \end{aligned} \quad (27)$$

with $s^\times(f, f'; g, g')$ a real inner product of $G \times G$ and with $\sigma^\times(f, f'; g, g')$ a symplectic form of $G \times G$ such that

$$(f, f'; g, g') = s^\times(f, f'; g, g') + i\sigma^\times(f, f'; g, g') \quad (28)$$

defines a positive-definite sesquilinear form of $G \times G$. These are analogues of the quasi-free states of Petz (1990). Interaction between e.m. field and current is supposed to be such that

$$\begin{aligned} \mathcal{F}^\times(f, 0; g, 0) &= \mathcal{F}(f, g) \\ \mathcal{F}^\times(0, f'; 0, g') &= \mathcal{F}^j(f', g') \end{aligned} \quad (29)$$

Let the GNS representation induced by $\mathcal{F}^\times(f, f'; g, g')$ satisfy

$$\mathcal{F}^\times(f, f'; g, g') = \langle \hat{W}^\times(g, g')^* \Omega \mid \hat{W}^\times(f, f')^* \Omega \rangle \quad (30)$$

Introduce generators $\hat{A}^l(f)$, $\hat{A}(f)$, and $\hat{A}^j(f)$ by

$$\begin{aligned} \hat{W}^\times(f, f) &= \hat{W}^l(f) = \exp(i\hat{A}^l(f)) \\ \hat{W}^\times(f, 0) &= \exp(i\hat{A}(f)) \\ \hat{W}^\times(0, f) &= \exp(i\hat{A}^j(f)) \end{aligned} \quad (31)$$

By construction is $\hat{A}^l(f) = \hat{A}(f) + \hat{A}^j(f)$. The commutation relations for the operators $\hat{W}^l(f)$ are

$$\hat{W}^l(f)\hat{W}^l(g) = e^{i\sigma^l(f, g)}\hat{W}^l(f + g) \quad (32)$$

The symplectic form appears in the r.h.s. of the commutation relations

$$\begin{aligned} [\hat{A}^l(f), \hat{A}^l(g)]_- &= -2i\sigma^\times(f, f; g, g) = -2i\sigma^l(f, g) \\ [\hat{A}(f), \hat{A}(g)]_- &= -2i\sigma^\times(f, 0; g, 0) \\ [\hat{A}^j(f), \hat{A}^j(g)]_- &= -2i\sigma^\times(0, f; 0, g) \\ [\hat{A}(f), \hat{A}^j(g)]_- &= -2i\sigma^\times(f, 0; 0, g) \end{aligned} \quad (33)$$

7. CLASSICAL CURRENTS

In the simplest case the currents $\hat{j}_\mu(q)$ are multiples of the identity operator. Then, the operators $\hat{W}^j(f')$ and $\hat{W}^j(g')$ can be taken out of the inner product of (24). A suitable guess is therefore

$$\mathcal{F}^j(f, g) = \exp(i A^{\text{cl}}(g - f)) \quad (34)$$

with

$$\hat{A}^{\text{cl}}(f) = - \int dq \int dq' f^\mu(q) \Delta_G(q' - q) j_\mu(q') \quad (35)$$

It is straightforward to verify that this function satisfies all requirements for being a correlation function.

Equation (34) allows very general classical potentials. Take, e.g., the Coulomb potential

$$A_\mu^{\text{cl}}(q) = \delta_{\mu,0} \frac{c}{|\mathbf{q}|} \quad (36)$$

where c is a constant, and where $q = (q_0, \mathbf{q})$. Eq. (35) is satisfied with external current

$$j_\mu(q) = \delta_{\mu,0} 4\pi c \delta^3(\mathbf{q}) \quad (37)$$

This means that a static charge of strength $4\pi c$ is located at the origin of space. The occurrence of a divergency in (36) does not produce any problem because it enters Eq. (34) in a form smeared out with test functions.

Note that $\sigma^j(f, g) = 0$. The commutation relations (33) suggest to take

$$\sigma^\times(f, 0; 0, g) = \sigma^\times(0, f'; 0, g) = 0 \quad (38)$$

It is still possible to include nontrivial correlations between the classical currents and the free-field operators by means of the inner product $s^\times(f, f'; g, g')$. However, standard results about quantized e.m. fields in the presence of classical currents are recovered when the choice

$$\begin{aligned} \sigma^\times(f, f'; g, g') &= \sigma(f, g) + A^{\text{cl}}(g - f) \\ s^\times(f, f'; g, g') &= s(f, g) \end{aligned} \quad (39)$$

is made

All together, the correlation function of the classical current model reads

$$\mathcal{F}^j(f, g) = \mathcal{F}(f, g) \mathcal{F}^j(f, g) \quad (40)$$

with $\mathcal{F}(f, g)$ given by (14) and $\mathcal{F}^j(f, g)$ given by (34). Such a simple product form reflects the known fact that the classical current model does not contain any interactions. By this is meant that in a Heisenberg picture the Hamiltonian is the sum of two free parts, without additional interaction term.

In the current model the interacting field operator equals the sum of the free-field operator and the classical potential generated by the external current

$$\hat{A}^1(f) = \hat{A}(f) + A^{\text{cl}}(f)\hat{\mathbb{1}} \tag{41}$$

This property is known in the literature—see e.g., Eq. (2.63) of Peskin and Schroeder (1995). Still, many handbooks use the classical current model to illustrate the scattering approach of QED, without mentioning (41). For sake of completeness we discuss some results of the scattering context in the next section.

Note that the field operators $\hat{A}^1(f)$ have some unusual properties. From (41), it is clear that they satisfy the same canonical commutation relations as the free-field operators. But the representation depends intrinsically on the details of the external current. Indeed, a shift in spacetime may map a nonzero field operator onto zero. This means that the shifted representation is not unitary equivalent with the original representation.

Let us analyze this point in somewhat more detail. A field operator $\hat{A}^1(f)$ vanishes if and only if $A^{\text{cl}}(f) = 0$ and $\langle \phi | \phi \rangle = 0$, where ϕ is the classical wave function associated with f . Now, if the current $j_\mu(q)$ is not trivial, then there exists a test function f for which $A^{\text{cl}}(f) \neq 0$ and $\langle \phi | \phi \rangle = 0$ holds. If the current is localized in part of spacetime then shift the test function with a vector a so that $\tau_a f$ vanishes in that part of spacetime where the current does not vanish. The result is that $A^{\text{cl}}(\tau_a f) = 0$. Because f and $\tau_a f$ determine the same classical wave function up to a phase factor one concludes that $\hat{A}^1(f) \neq 0$ while $\hat{A}^1(\tau_a f) = 0$.

8. RADIATION FIELDS

Let us first verify what happens if $A^{\text{cl}}(f)$ is a solution of the homogeneous d'Alembert equation, i.e., the current vanishes. Then, one can write

$$\begin{aligned} A^{\text{cl}}(f) &= \int_{\mathbb{R}^3} d\mathbf{k} \frac{1}{2|\mathbf{k}|} (a^\mu(\mathbf{k})\overline{\phi_\mu(\mathbf{k})} + \overline{a^\mu(\mathbf{k})}\phi_\mu(\mathbf{k})) \\ &= -2 \text{Re}\langle a | \phi \rangle \end{aligned} \tag{42}$$

with ϕ the classical wave function determined by the test functions f and with

$$a_\mu(\mathbf{k}) = \frac{1}{\sqrt{2\pi}} \tilde{A}_\mu^{\text{cl}}(|\mathbf{k}|, \mathbf{k}) \tag{43}$$

Hence, the correlation function can be written as

$$\begin{aligned} \mathcal{F}^1(f, g) &= \mathcal{F}(f, g) \exp(-2i \text{Re}\langle a | \psi - \phi \rangle) \\ &= \exp(i \text{Im}\langle \phi + 2ia | \psi + 2ia \rangle) \exp(-(1/2)\langle \phi - \psi | \phi - \psi \rangle) \end{aligned} \tag{44}$$

In this expression the Fourier-transformed classical potential a , when multiplied with $2i$, behaves as a Fourier-transformed test function. This relation is a duality

between test functions and fields and has been studied in Kuna and Naudts (2002). In particular, the definition of field operators $\hat{A}(f)$ can be extended to complex arguments. Hence, one can write

$$\begin{aligned} \mathcal{F}^1(f, g) &= \mathcal{F}(f + i\pi^{-1}A^{cl}, g + i\pi^{-1}A^{cl}) \\ &= \langle \hat{W}(g + i\pi^{-1}A^{cl})^* \Omega \mid \hat{W}(f + i\pi^{-1}A^{cl})^* \Omega \rangle \\ &= \langle \Omega(A^{cl}) \mid \hat{W}(g)\hat{W}(f)^* \Omega(A^{cl}) \rangle \end{aligned} \tag{45}$$

with

$$\Omega(A^{cl}) - \hat{W}(i(2\pi)^{-1}A^{cl})^* \Omega \tag{46}$$

To obtain the latter use that $\hat{W}(f + 2g) = \hat{W}(g)\hat{W}(f)\hat{W}(g)$. This shows that in this case the correlation functions (40) are those of a coherent state, as expected from conventional photon theory.

Next let us make a link with the scattering approach. Consider a classical field $A_{\mu}^{cl}(q)$ which vanishes for very negative times $q_0 \ll 0$. This implies that $j_{\mu}(q) = 0$ for very negative times. Assume that $j_{\mu}(q) = 0$ holds also for very positive times $q_0 \gg 0$. Then, $A_{\mu}^{cl}(q)$ for $q_0 \gg 0$ is the radiation field produced by currents $j_{\mu}(q)$ which are only active during a finite interval of time q_0 . This radiation field can be expressed in terms of the current using the Feynman’s propagator

$$D_F(q) = \frac{1}{(2\pi)^4} \int_{\mathbb{R}^4} dk \frac{1}{k^{\nu}k_{\nu}} e^{-ik^{\mu}q_{\mu}} \tag{47}$$

which is a Green’s function of the d’Alembert equation. One finds

$$\begin{aligned} A_{\mu}^{cl}(q) &= A_{\mu}^{hom}(q) - \int dq' j_{\mu}(q') D_F(q - q') \\ &= A_{\mu}^{hom}(q) - \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} dk \frac{1}{k^{\nu}k_{\nu}} e^{-ik^{\lambda}q_{\lambda}} \tilde{j}_{\mu}(k) \end{aligned} \tag{48}$$

where $A_{\mu}^{hom}(q)$ is a solution of the homogeneous d’Alembert equation. Because of the assumption that $A_{\mu}^{cl}(q) = 0$ when $q_0 \ll 0$, one must have

$$A_{\mu}^{hom}(q) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} dk \frac{1}{k^{\nu}k_{\nu}} e^{-ik^{\lambda}q_{\lambda}} \tilde{j}_{\mu}(k), \quad q_0 \ll 0 \tag{49}$$

The latter equation can be written as

$$A_{\mu}^{hom}(q) = -\frac{1}{(2\pi)} \text{Im} \int_{\mathbb{R}^3} dk \frac{1}{2|\mathbf{k}|} e^{i|\mathbf{k}|q_0} e^{i\mathbf{k}\cdot\mathbf{q}} \tilde{j}_{\mu}(|\mathbf{k}|, \mathbf{k}) \tag{50}$$

In this form the expression is valid for all q . Indeed, one checks immediately that this is a solution of the homogeneous d’Alembert equation. For $q_0 \ll 0$ one finds $A_{\mu}^{cl}(q) = 0$ by construction, for $q_0 \gg 0$ one has $A_{\mu}^{cl}(q) = 2A_{\mu}^{hom}(q)$.

Smearing out $A_\mu^{\text{hom}}(q)$ with a test function f one obtains (see (42))

$$A^{\text{hom}}(f) = \frac{1}{2\pi} \text{Im}\langle\phi | a\rangle \tag{51}$$

with ϕ the classical wave function determined by the test functions f , and with

$$a_\mu(\mathbf{k}) = \frac{1}{\sqrt{2\pi}} \tilde{j}_\mu(|\mathbf{k}|, \mathbf{k}) \tag{52}$$

Hence, for test functions with support lying far in the future one has

$$A^{\text{cl}}(f) = \frac{1}{\pi} \text{Im}\langle\phi | a\rangle \tag{53}$$

The correlation function (40) for a pair of such functions reads

$$\mathcal{F}^1(f, g) = \mathcal{F}(f, g) \exp((i/\pi) \text{Im}\langle\psi - \phi | a\rangle) \tag{54}$$

Compare this with

$$\begin{aligned} &\langle \hat{W}(g)^* \hat{W}(v)^* \Omega | \hat{W}(f)^* \hat{W}(v)^* \Omega \rangle \\ &= \exp(i \text{Im}\langle\phi - \psi | \xi\rangle) \langle \hat{W}(v + g)^* \Omega | \hat{W}(v + f)^* \Omega \rangle \\ &= \exp(i \text{Im}\langle\phi - \psi | \xi\rangle) \mathcal{F}(v + f, v + g) \\ &= \exp(2i \text{Im}\langle\phi - \psi | \xi\rangle) \mathcal{F}(f, g) \end{aligned} \tag{55}$$

where ξ is the classical wave function corresponding with the test function v . The two expressions coincide provided there exists a test function v such that $\xi = -a/2\pi$ holds. If this is the case, then the radiation field is described by the coherent state with wave vector $\hat{W}(v)^* \Omega$. This coincides with the standard result that, up to a phase factor, the S-matrix is a displacement operator, and the radiative field is a coherent state—see e.g., (Białynicki-Birula and Białynicki-Birula, 1975), Section 13. The problem is not the appearance of a complex test function, which has been explained above, but the possibility of an infrared divergency. Indeed, a test function v , such that $\xi = -a/2\pi$ holds, will not always exist. One can of course try to approximate $a_\mu(\mathbf{k}) = \tilde{j}_\mu(|\mathbf{k}|, \mathbf{k})/\sqrt{2\pi}$ by classical wave functions ξ_n . However, this will work only if

$$\langle a | a \rangle = -\frac{1}{2\pi} \int_{\mathbb{R}^3} d\mathbf{k} \frac{1}{2|\mathbf{k}|} \overline{\tilde{j}^\mu(|\mathbf{k}|, \mathbf{k})} \tilde{j}_\mu(|\mathbf{k}|, \mathbf{k}) \tag{56}$$

is finite. This is precisely the condition for absence of infrared divergency found in (Białynicki-Birula and Białynicki-Birula, 1975), Section 13, in case of an example. For recent progress on the infrared divergency problem in the context of Nelson’s model see Arai (2001) and Lörinczi, Minlos, and Spohn (2002).

9. QUANTUM CURRENTS

Let us now consider a simplified model of quantum currents. Start with creation and annihilation operators \hat{b}^* and \hat{b} of a harmonic oscillator. They satisfy the canonical commutation relations

$$[\hat{b}, \hat{b}^*]_- = 1 \tag{57}$$

Let the given complex functions $\alpha_\mu(q)$ satisfy the continuity equation

$$\partial_\mu \alpha^\mu(q) = 0 \tag{58}$$

They are used to define currents $\tilde{j}_\mu(q)$ by the relation

$$\hat{j}_\mu(q) = \alpha_\mu(q)\hat{b}^* + \overline{\alpha_\mu(q)}\hat{b} \tag{59}$$

The smeared-out potentials (19) become

$$\hat{A}^j(f) = y(f)\hat{b}^* + \overline{y(f)}\hat{b} \tag{60}$$

with

$$y(f) = - \int dq \int dq' f^\mu(q)\Delta_G(q' - q)\alpha_\mu(q') \tag{61}$$

They satisfy commutation relations

$$[\hat{A}^j(f), \hat{A}^j(g)]_- = -2i\sigma^j(f; g) \tag{62}$$

with

$$\sigma^j(f, g) = \text{Im}(\overline{y(f)}y(g)) \tag{63}$$

Let Ω denote the ground state of the harmonic oscillator. It satisfies $b\Omega = 0$ and determines the correlation function $\mathcal{F}^j(f, g)$ via (24). One obtains by means of standard calculation

$$\mathcal{F}^j(f, g) = \exp(-i\sigma^j(f, g)) \exp(-(1/2)s^j(f - g, f - g)) \tag{64}$$

with

$$s^j(f, g) = \frac{1}{2}(y(f)\overline{y(g)} + y(g)\overline{y(f)}) \tag{65}$$

In this state the quantum expectation of the current $\hat{j}_\mu(q)$ vanishes. The second moments equals

$$\langle \Omega | \hat{j}_\mu(q)\hat{j}_\nu(q')\Omega \rangle = \overline{\alpha_\mu(q)}\alpha_\nu(q') \tag{66}$$

The interaction between photons and currents is modeled by assuming the existence of a real-linear function $x(f)$ such that

$$[b, \hat{A}(f)]_- = x(f)\mathbb{I} \tag{67}$$

This implies

$$\begin{aligned} \sigma^\times(f, 0; 0, g') &= \frac{i}{2}[\hat{A}(f), \hat{A}^j(g')]_- \\ &= \text{Im}(x(f)\overline{y(g')}) \end{aligned} \tag{68}$$

The obvious choice of symplectic form $\sigma^\times(f, f'; g, g')$ is then

$$\begin{aligned} \sigma^\times(f, f'; g, g') &= \sigma(f, g) + \sigma^j(f', g') \\ &\quad + \text{Im}(x(f)\overline{y(g')}) + \text{Im}(y(f')\overline{x(g)}) \end{aligned} \tag{69}$$

The positivity requirement (28) suggest now to define

$$\begin{aligned} s^\times(f, f'; g, g') &= s(f, g) + s^j(f', g') \\ &\quad + \text{Re}(x(f)\overline{y(g')}) + \text{Re}(y(f')\overline{x(g)}) \end{aligned} \tag{70}$$

Positivity is satisfied provided

$$|x(f)|^2 \leq s(f, f) = \langle \phi | \phi \rangle \tag{71}$$

This implies the existence of functions $f_x^{(1)}$ and $f_x^{(2)}$ satisfying

$$s(f_x^{(1)}, f_x^{(1)}) + s(f_x^{(2)}, f_x^{(2)}) \leq 1 \tag{72}$$

for which

$$x(f) = s(f_x^{(1)}, f) + is(f_x^{(2)}, f) \tag{73}$$

The symplectic form $\sigma^\times(f, f'; g, g')$ and the bilinear form $s^\times(f, f'; g, g')$ together determine the correlation function $\mathcal{F}^\times(f, f'; g, g')$ via (27), and a corresponding state of the covariance system with group $G \times G$. The diagonal $\mathcal{F}^1(f, g)$ described a state of the quantized e.m. field interacting with a quantum current. It is the latter state which is analyzed below.

10. A QUANTUM SOURCE OF E.M. RADIATION

First of all note that state determined by $\mathcal{F}^1(f, g)$ from a classical point of view described always a vacuum. Indeed, from

$$\mathcal{F}^1(f, 0) = \mathcal{F}(f, 0)e^{(1/2)|x(f)|^2} e^{-(1/2)|x(f)-y(f)|^2} \tag{74}$$

follows by expansion to first order in f that

$$\langle A^1(f) \rangle = 0 \tag{75}$$

The latter quantity is the classical part of the smeared-out e.m. vector potential. That it vanishes is in agreement with the pure quantum nature of the currents $\hat{j}_\mu(q)$ whose quantum expectation vanishes as well.

Next consider the scattering situation with a current localized in space-time, i.e., $\alpha_\mu(q) = 0$ outside some bounded region in the vicinity of the origin of space-time. In addition, let the Green's function $\Delta_G(q)$ in (61) be the retarded Green's function. Then $y(f) = 0$ holds for all f with support in the far past. As a consequence, for f', g' with support in the far past is $\mathcal{F}^\times(f, f'; g, g') = \mathcal{F}(f, g)$. Hence, the state of the systems in the past is the vacuum of the free e.m. field.

The function $y(f)$ does not vanish for all f with support in the future, even when $\alpha_\mu(q)$ is again zero. In other words, the quantum current produces a radiation field. A first observation is that this radiation field cannot be coherent because the square of $y(f)$ appears in (64). This is also obvious from (75). Nontrivial quantum fluctuations are present, as can be seen by expanding $\mathcal{F}^I(f, g)$ to first order in f and g

$$\langle \hat{A}^I(g)\hat{A}^I(f) \rangle = \langle \phi | \psi \rangle - \overline{x(f)}y(g) - \overline{y(f)}x(g) + \overline{y(f)}y(g) \tag{76}$$

The first term in the r.h.s. are the vacuum fluctuations. The last term describes field fluctuations which are identical to those of a classical radiation field. This leads to the remarkable observation that in this model quantum currents produce e.m. fields which propagate like classical radiation fields. They differ from them because their quantum expectation vanishes. Whether such quantum radiation fields exist in nature is not immediately clear. In more sophisticated models one can expect that these quantum radiation fields will survive in some sense. Indeed, any quantum current $\hat{j}_\mu(q)$ can be decomposed into the sum of a classical current $\langle \hat{j}_\mu(q) \rangle$ and a remainder with vanishing average. Now assume that the interacting field operators $\hat{A}_\mu^I(q)$ satisfy the d'Alembert equation. Then, by linearity, the field operators decompose into the sum of a field produced by the classical current and a remainder without classical analogue, of the type found in the present model.

A limitation of the model is that the quantum current is described by one single harmonic oscillator. As a consequence, the interacting field operators $\hat{A}^I(f)$ satisfy unsatisfactory commutation relations (see (33) and (69))

$$[\hat{A}^I(f), \hat{A}^I(g)]_- = 2i\sigma^I(f, g) \tag{77}$$

with

$$\begin{aligned} \sigma^I(f, g) = & \sigma(f, g) - \text{Im } x(f)\overline{x(f)} \\ & + \text{Im}(x(f) + y(f))\overline{x(g) + y(g)} \end{aligned} \tag{78}$$

It is tempting to modify the model in such a way that the first two contributions to $\sigma^I(f, g)$ cancel. One can hope that in such a modified model the energy density of the vacuum, which is infinite in the absence of interactions, becomes finite in presence of quantum fields produced by quantum currents.

11. CONCLUSIONS

The present paper develops a rigorous theory for quantized e.m. fields interacting with given current fields. The formalism of covariance systems is used. The main tool in this approach is the vacuum-to-vacuum correlation function. It is not assumed *a priori* that the field operators are those of the noninteracting theory. Instead, they are the sum of the free-field operator and of a field operator, which is the solution of the d'Alembert equation with the given quantum current as source term.

The theory has been applied to two simple models, the first of which is the well-known model of quantized e.m. fields interacting with a classical current. We show that our approach agrees with the standard results. In case the currents vanish outside a finite part of spacetime then our result describes radiation fields which correspond with coherent states, at least, if no infrared divergency occurs. The second model describes a current whose quantum expectation vanishes. It produces a radiation field whose quantum expectation vanishes as well. More sophisticated models are expected to produce similar results.

Our main conclusion is that the algebra of field operators depends on details of the applied current. In this aspect our work goes beyond other approaches based on fixed algebraic structures and their representations. The emphasis on correlation functions solves the related technical problems in an elegant way.

Field operators in the present paper are smeared out with test functions over spacetime, and *not* with test functions over three-dimensional space, as is often done. We were not able to make a transition between these two approaches. In particular, we did not obtain a Heisenberg picture with a Hamiltonian, dependent on the external current, describing the time evolution of field operators smeared out with test functions over three-dimensional space. If it turns out that such description does not exist, then this is bad news for the standard approach, based on scattering theory, which takes the existence of an interaction picture for granted.

The present work opens perspectives which may eventually lead to a rigorous formalism of quantum electrodynamics. The next step to take along the lines of the present paper is a rigorous description of a field of Dirac electrons interacting with a classical e.m. field. The algebra of Dirac currents is more complicated than what is supported by the present paper. The resulting technical problems have to be solved as well.

APPENDIX A

Here, we give a justification for assumption (3).

Assume two vector potentials differ only by a gauge transformation

$$A'_\mu(q) = A_\mu(q) + \partial_\mu \mathcal{X}(q) \tag{A.1}$$

with $\mathcal{X}(q)$ an arbitrary solution of the d'Alembert equation. Then, the smeared-out

fields satisfy

$$\begin{aligned}
 A'(f) - A(f) &= \int_{\mathbb{R}^4} dq f^\mu(q) \partial_\mu \mathcal{X}(q) \\
 &= i \int_{\mathbb{R}^4} dk \tilde{\mathcal{X}}(-k) \tilde{f}^\mu(k) k_\mu
 \end{aligned}
 \tag{A.2}$$

Since the two vector potentials are physically equivalent, it should not be possible to distinguish them by means of the test function f . The condition $A'(f) = A(f)$ implies then that (3) must hold for all wave vectors k satisfying $k^\mu k_\mu = 0$.

Assume now that $\tilde{g}_\mu(k)$ satisfies $k^\mu \tilde{g}_\mu(k) = 0$ whenever $k^\nu k_\nu = 0$, in such a way that the function

$$\mathcal{X}(k) = \frac{1}{k^\nu k_\nu} k^\mu \tilde{g}_\mu(k)
 \tag{A.3}$$

remains continuous. Then, one can decompose $\tilde{g}_\mu(k)$ into parts parallel and orthogonal to the wave vector

$$\tilde{g}_\mu(k) = \tilde{f}_\mu(k) + k_\mu \tilde{\mathcal{X}}(k)
 \tag{A.4}$$

where $\tilde{f}_\mu(k)$ satisfies (3) for all k . One verifies immediately that $A(g) = A(f)$ holds for any vector potential A . Hence, we can always chose the test functions f , satisfying (3) for all k , as representative for a whole class of equivalent test functions g , satisfying (3) when $k^\mu k_\mu = 0$, and such that (A.3) remains continuous.

APPENDIX B

Here, we show that the correlation function (14) satisfies the necessary conditions. The extension of the arguments to correlation function (40) is straightforward.

Normalization $\mathcal{F}(0, 0) = 1$ is clear. Positivity follows from

$$\begin{aligned}
 \sum_{mn} \bar{\lambda}_n \lambda_n \mathcal{F}(f_m, f_n) &= \sum_{mn} \bar{\lambda}_n \lambda_n e^{\langle f_m | f_n \rangle} e^{-(1/2)\langle f_m | f_m \rangle} e^{-(1/2)\langle f_n | f_n \rangle} \\
 &= \sum_{mn} \bar{\mu}_n \mu_n e^{\langle f_m | f_n \rangle}
 \end{aligned}
 \tag{B.1}$$

with

$$\mu_m = \lambda_m e^{-(1/2)\langle f_m | f_m \rangle}
 \tag{B.2}$$

Note that the matrix with elements $\langle f_m | f_n \rangle$ is positive definite. Hence, positivity of (B.1) follows by means of Schur's lemma.

Finally, covariance follows from

$$\begin{aligned}\mathcal{F}(f+h, g+h) &= e^{i \operatorname{Im}\langle f+h|g+h\rangle} e^{-(1/2)\langle f-g|f-g\rangle} \\ &= e^{i \operatorname{Im}\langle h|g\rangle} e^{i \operatorname{Im}\langle f|h\rangle} \mathcal{F}(f, g)\end{aligned}\tag{B.3}$$

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REFERENCES

- Arai, A. (2001). Ground state of the massless Nelson model without infrared cutoff in a non-Fock representation. *Rev. Math. Phys.* **13**(9), 1075–1094.
- Białynicki-Birula, I. and Białynicki-Birula, Z. (1975). *Quantum Electrodynamics*, Pergamon Press, Oxford.
- Carey, A. L., Gaffney, J. M., and Hurst, C. A. (1977). A C^* -algebra formulation of the quantization of the electromagnetic fields. *J. Math. Phys.* **18**, 629–640.
- Kibble, T. W. B. (1968). Coherent soft-photon states and infrared divergencies. *I. Classical Currents*, *J. Math. Phys.* **9**(2), 315–324.
- Kuna, M. and Naudts, J. (2002). Covariance approach to the free photo field. In *Probing the Structure of Quantum Mechanics: Nonlinearity, Nonlocality, Computation and Axiomatics*, D. Aerts, M. Czachor, and T. Durt, eds., World Scientific, Singapore, pp. 368–393.
- Lőrinczi, J., Minlos, R. A., and Spohn, H. (2002). Infrared regular representation of the three dimensional massless Nelson model. *Lett. Math. Phys.* **59**(3), 189–198.
- Naudts, J. and Kuna, M. (2001). Covariance systems. *J. Phys. A*, **34**, 9265–9280.
- Petz, D. (1990). *An Invitation to the Algebra of Canonical Commutation Relations*, Leuven University Press, Leuven.
- Peskin, M. E. and Schroeder, D. V. (1995). *An Introduction to Quantum Field Theory*, Addison-Wesley, Reading, MA.